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SOME WEAKLY NONLINEAR AMPLITUDE EQUATIONS DESCRIBING THE BEHAVIOR OF A THIN LAYER IN A TWO-PHASE FLOW OF VISCOUS HEAT-CONDUCTING LIQUIDS ALONG A CYLINDER

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1. Introduction. A weakly nonlinear stability of an interface of initially cylindrical shape has been studied in [1–3] for a two-phase laminar flow of immiscible viscous incompressible liquids along a hollow pipe. One of the phases (named the core) is separated from the surface of the pipe by a thin film layer of the second liquid. The character of stability for this class of two-phase flows is important for some natural and technological processes. Some possible areas of application are as follows.

(1) Displacement of one liquid by another in capillaries (for example, upon washing out of oil from rock strata), when a part of the liquid phase with a stronger wettability of the capillary-channel surface remains on the walls in the form of thin films. It surrounds the other liquid which is in the center in the form of threads separated by the first-phase gaps [3]. In the central part of these regions, the motion resembles locally the laminar flow of a liquid core surrounded by a thin film. It is important to take into account the problem of stability of such idealized flows in analyzing the processes of displacement of liquids and their two-phase motion in capillaries, cylindrical channels, and in slots of porous media: film instability and decay can substantially slow down liquid flow [3].

(2) Organization of a thin "lubrication" layer in pipes to decrease energy losses in transfer of liquid products.

(3) Realization of special flows of the above class to apply a thin coating to cylindrical surfaces.

The use of the long-wave approximation of the boundary-layer theory is a productive method to investigate the weakly nonlinear stability of film flows. Using this and Stokes approximations, Frenkel et al. [1] have shown that when the dynamic viscosities of liquids are equal and some additional conditions are satisfied, the dynamics of perturbations in the thin layer does not depend on the evolution of fluctuations in the core, and the behavior of the interface is described by the Kuramoto–Sivashinskii equation:

$$\eta_\tau + N\eta\eta_z + U\eta_{zz} + S\eta_{zzzz} = 0, \qquad (1.1)$$

where η is the deviation of the boundary from the equilibrium state, τ is time, z is the spatial variable, and the corresponding subscripts are used to denote partial derivatives.

The influence of viscous stratification that can lead to conjugation of perturbations in the core and in the film has been investigated in [2]. This gives rise to the occurrence of an integral term in the amplitude equation (1.1) (in this case, the asymptotic problem in the core is constructed on the basis of a special approximation and is solved with the Fourier transform).

In the present paper, on the basis of the approaches used in [1, 2], we consider a modification of the corresponding amplitude equations if thermodynamic effects are taken into account.

2. Formulation of the Problem. Let $\hat{r}$ and $\hat{z}$ be the radial and axial cylindrical coordinates. We consider the initial motion of two immiscible liquids flowing in the regions $\{\hat{r} < \hat{b}, -\infty < \hat{z} < \infty\}$ and $\{\hat{b} < \hat{r} < \hat{a}, -\infty < \hat{z} < \infty\}$ inside a cylindrical surface S which is specified by the equation $\hat{r} = \hat{a}$. Both liquids are considered viscous, heat-conducting, and incompressible. The motion is assumed to be rotationally

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symmetric. The effect of gravity is not taken into account (which, for example, can be justified under certain conditions if the cylinder S is a fairly narrow capillary [4]). It is assumed that $(\hat{a} - \hat{b})/\hat{b} \equiv \varepsilon \ll 1$.

In the general case, the interface Γ is given by the equation $\hat{r} = \hat{L}(\hat{z}, \hat{t})$, and the liquids of the core and of the film occupy, respectively, the regions $\Omega_1 = \{\hat{r} < \hat{L}(\hat{z}, \hat{t})\}$ and $\Omega_2 = \{\hat{L}(\hat{z}, \hat{t}) < \hat{r} < \hat{a}\}$. In what follows, the region Ω_1 is called the core, and the area Ω_2 is called the thin layer or film, and the quantities related to the regions are denoted by the subscripts 1 and 2, respectively.

Let $\hat{U}_i$ and $\hat{W}_i$ be the radial and axial components of the velocity vector, $\hat{P}_i$ be the pressure, and $\hat{\Theta}_i$ be the temperature of the liquid in the region Ω_i (hereafter $i = 1, 2$).

The following parameters are used in the problem: σ is the coefficient of surface tension, which is assumed to be a linear function of temperature [$\sigma = \sigma_* - æ(\hat{\Theta} - \hat{\Theta}_*)$, where $æ > 0$ is a constant], ρ_i are the densities, ν_i are the kinematic viscosities, μ_i are the dynamic viscosities, χ_i are the thermal diffusivities, c_i are the specific heats, and $k_i = \rho_i c_i \chi_i$ are the thermal conductivities of the liquids.

The basic flow, whose stability is investigated, is induced by a constant pressure gradient along the cylinder and has the form $\hat{U}_{0i} = 0$, $\hat{W}_{0i} = A_i\hat{r}^2 + B_i$, $\hat{\Theta}_{0i} = K_{1i}\hat{r}^4 + K_{2i}\ln(\hat{r}) + K_{3i}$, and $\hat{P}_{0iz} = -\hat{F}$, $\hat{F} = \mathrm{const} > 0$. The wall of S has a constant temperature $\hat{\Theta}_S$.

We chose the following scale factors for the nondimensional problem: $\hat{b}$ for spatial variables, $\hat{W} = \hat{W}_{01}(0) = \hat{F}\hat{b}^2(1 + (\mu_1/\mu_2)(\hat{a}^2\hat{b}^2 - 1))/4\mu_1$ for velocity, $\hat{b}/\hat{W}$ for time, $\hat{P} = \rho_1\hat{W}^2$ for pressure, and $\hat{\Theta} = \hat{\Theta}_S - \hat{\Theta}_{01}(0)$ for temperature. Below, the desired functions and variables are considered nondimensional (the superscript hat is omitted). In what follows, $\hat{a}/\hat{b} = a$ and $\hat{L}/\hat{b} = L$.

The motion in the regions is described by the Navier–Stokes, continuity, and thermal-conductivity equations (for simplicity, the subscripts are omitted):

$$U_t + UU_r + WU_z = -(\rho_1/\rho)P_r + (1/\mathrm{Re})(\Delta U - (1/r^2)U); \tag{2.1}$$

$$W_t + UW_r + WW_z = -(\rho_1/\rho)P_z + (1/\mathrm{Re})\Delta W; \tag{2.2}$$

$$U_r + (1/r)U + W_z = 0; \tag{2.3}$$

$$\Theta_t + U\Theta_r + W\Theta_z = (1/\mathrm{Pe})\Delta\Theta + 2\mathrm{Dis}\{U_r^2 + (1/r^2)U^2 + W_z^2 + (W_r + U_z)^2/2\}. \tag{2.4}$$

Here $\Delta \equiv \partial^2/\partial r^2 + \partial^2/\partial z^2 + (1/r)\partial/\partial r$.

The boundary conditions have the form [5]

for $r = a$

$$U_2 = 0, \quad W_2 = 0, \quad \Theta_2 = \Theta_S \quad \text{for } r = a; \tag{2.5}$$

for $r = 0$ all the quantities are bounded;

for $r = L(z, t)$

$$[U] = [W] = [\Theta] = 0; \tag{2.6}$$

$$-\mathrm{Re}_1[P] + 2(1 + L_z^2)^{-1}([MU_r] - L_z[M(W_r + U_z)] + L_z^2[MW_z])$$
$$= (\mathrm{We} + \mathrm{Mn}(\Theta_2 - \Theta_*))(-1 + LL_{zz} - L_z^2)L^{-1}(1 + L_z^2)^{-3/2}; \tag{2.7}$$

$$(1 + L_z^2)^{-1/2}(2L_z[MU_r] + (1 - L_z^2)[M(W_r + U_z)] - 2L_z[MW_z]) = \mathrm{Mn}(\Theta_r L_z + \Theta_{2z}); \tag{2.8}$$

$$[Q(\Theta_r - L_z\Theta_z)] = \mathrm{Es}\,\Theta_2(1 + L_z^2)^{-1/2}(U_r - U_zL_z - W_rL_z + W_zL_z^2); \tag{2.9}$$

$$U_2 = L_t + W_2L_z, \tag{2.10}$$

where $[(.)] \equiv (.)_1 - (.)_2$, $\mathrm{Re}_i = \hat{W}\hat{b}/\nu_i$, $\mathrm{Pe}_i = \hat{W}\hat{b}/\chi_i$, $\mathrm{Dis}_i = (\nu_i\hat{W})/(c_i\hat{\Theta}\hat{b})$, $\mathrm{We} = \sigma_*/(\mu_1\hat{W})$, $\mathrm{Mn} = -æ\hat{\Theta}/(\mu_1\hat{W})$, $\mathrm{Es} = æ\hat{W}/k_1$, $M_i = \mu_i/\mu_1$, $Q_i = k_i/k_1$, $\Theta_* = \hat{\Theta}_*/\hat{\Theta}$, and $\Theta_S = \hat{\Theta}_S/\hat{\Theta}$. From now on, $m = M_2$ and $q = Q_2$.

In view of the continuity of condition (2.6), the subscript 2 is introduced (if possible) on the right-hand sides of conditions (2.7)–(2.9) and also in (2.10). Conditions (2.7) and (2.8) express the stress balance at the interface, condition (2.9) implies equilibrium between the heat-flux jump at Γ and the variation in the internal energy of this surface, which is associated with variation in the boundary area [5], and (2.10) is a kinematic condition.

Note that there is a quantity $\Theta_r(r = L)$ on the right-hand side of (2.8). Its value is a certain average between $\Theta_{1r}(r = L)$ and $\Theta_{2r}(r = L)$ and depends on the concrete problem. A similar remark is also valid for the derivatives $U_r(r = L)$ and $W_r(r = L)$ on the right-hand side of (2.9).

The initial flow, whose stability is studied here, is described by the following solution of problem (2.1)-(2.10):

$$\begin{gathered} L = 1, \quad U_{0i} \equiv 0, \quad W_{01} = 1 - mr^2/(a^2 + m - 1), \quad W_{02} = (a^2 - r^2)/(a^2 + m - 1), \\ P_{0iz} = -\hat{F}\hat{b}/\hat{P} = \text{const}, \quad \Theta_{0i} = K_{1i}r^4 + K_{2i}\ln(r) + K_{3i}, \\ K_{11} = qm/D, \quad K_{21} = 0, \quad K_{31} = \Theta_S - 1, \quad K_{12} = 1/D, \quad K_{22} = 4(m-1)/D, \\ K_{32} = \Theta_S - K_{12}a^4 - K_{22}\ln(a) \quad (D = a^4 - 1 + qm + 4(m-1)\ln(a)). \end{gathered} \tag{2.11}$$

We denote the small nondimensional width of the film by ε ($\varepsilon \equiv a - 1 \ll 1$).

3. Estimation of the Orders of Perturbations. We assume for the perturbed motion that $U_i = U_{0i} + u_i$, $W_i = W_{0i} + w_i$, $P_i = P_{0i} + p_i$, $\Theta_i = \Theta_{0i} + \theta_i$, and $L = 1 + l$.

Our goal is to derive an amplitude equation describing the wave motion at the interface with a characteristic spatial scale of the order of the core-region radius. Therefore, in accordance with the long-wave approximation of the boundary-layer theory, we introduce a new radial coordinate into the film: $r = 1 + \varepsilon - \varepsilon y$, $0 \leqslant y \leqslant 1$.

Let us determine the orders of some parameters of the problem. Let

$$\text{We} = O(\varepsilon^{-1}), \quad \rho_2/\rho_1 = O(1), \quad m = \mu_2/\mu_1 = O(1), \quad q = k_2/k_1 = O(\varepsilon).$$

We consider the development of perturbations at a stage at which the deviation of the interface Γ from its unperturbed state has the order $\delta(l = \delta\eta(z,t))$, with $\delta \ll \varepsilon$. An equation describing the weakly nonlinear stage of evolution of the interface is derived on the basis of the kinematic condition (2.10). This condition is a prototype of the amplitude equation, whose final form depends on the adopted assumptions on the physical organization of fluctuation development. We do not deal with a linear analysis of stability, assuming that the final perturbations appear in a definite way in the system at a certain stage (for example, they are introduced directly from the outside).

Having obtained the asymptotic linear problem, whose solution is expressed in terms of functionals of the function η, we reduce the initial system (2.1)-(2.9). We then use the solution obtained by replacing the velocity components appearing in the reduced kinematic condition (2.10) by their expressions in terms of η. In the coordinate system moving along the z axis with velocity $W_{02}(1)$, the influence of the basic flow $[W_{02}(1 + l)]$ on the form of the kinematic condition leads to the occurrence of the quadratic nonlinearity in the expansion of (2.10) at the unperturbed boundary.

Let us consider the following variant of the development of fluctuations. In the boundary-layer approximation, with a sufficiently strong surface tension, a deformation of the interface usually induces a large pressure perturbation in the thin layer. This pressure perturbation prescribes the scales of the other hydrodynamic variables, generating, in particular, a semiparabolic profile of longitudinal-velocity perturbation. We assume that

$$p_2 \sim (\text{We}/\text{Re}_1)\delta \quad [(\text{Re}_1 p_2 \sim \text{We}(l + l_{zz})]; \tag{3.1}$$

$$w_2 \sim \text{Re}_2\, \varepsilon^2 p_2 \sim \text{We}\, \varepsilon^2\delta \quad [\text{from the Navier–Stokes equation (2.2)}]; \tag{3.2}$$

$$u_2 \sim \varepsilon w_2 \sim \text{We}\, \varepsilon^3\delta \quad [\text{from the continuity equation (2.3)}]. \tag{3.3}$$

Owing to the viscous stratification, $W_{02}(1 + \delta\eta) - W_{01}(1 + \delta\eta) = 2(1 - 1/m)\delta\eta + O(\delta\varepsilon) = O(\delta)$, and, since $w_2 < O(\delta)$, from the condition of velocity continuity at the interface we obtain

$$w_1 = O(\delta). \tag{3.4}$$

It then follows from (2.2) and (2.3) that

$$u_1 = O(\delta); \tag{3.5}$$

$$p_1 = O(\delta/\mathrm{Re}_1). \tag{3.6}$$

In accordance with (3.2)–(3.5), we have the following estimates for the radial derivatives on the right-hand side of (2.9):

$$w_r(L) = O(\delta); \tag{3.7}$$

$$O(\mathrm{We}\,\varepsilon^2\,\delta) < u_r(L) < O(\delta), \tag{3.8}$$

and, according to (2.11),

$$W_{0r}(L) = O(1). \tag{3.9}$$

Let us consider a situation in which in the process of energy transfer [Eq. (2.9)] small variations in the internal energy of the interface Γ, which are associated with variations in the area of this interface surface, are linked with the temperature perturbation in the film.

We first note that, in accordance with (2.11), $\Theta_{02}(1) = T + O(\varepsilon)$, where $T = \Theta_{S2} - \{(a+1)(a^2+1) + 4(m-1)\}/\{(a+1)(a^2+1) + qm + 4(m-1)\}$, and assume that $T = O(1)$. Then, taking into account the estimates (3.7)–(3.9), we assume that

$$\theta_2 \sim \mathrm{Es}\,TW_{0r}(1)\varepsilon\delta/q \sim \mathrm{Es}\,\delta. \tag{3.10}$$

We consider the case $\mathrm{Mn} \sim \mathrm{We}\,\varepsilon/\mathrm{Es}$ to relate the quantity θ_2 to hydrodynamic perturbations [in this case, by condition (2.8)] and, hence, to make it possible for the dynamics of temperature perturbations to influence the evolution of the interface Γ.

Since $q = O(\varepsilon)$, we have, owing to (2.11), $\Theta_{01r} = O(1)$ and $\Theta_{02r} = O(\varepsilon^{-1})$. Therefore, $\Theta_{01}(1+\delta\eta) - \Theta_{02}(1+\delta\eta) = O(\delta/\varepsilon)$. If the orders of the parameters are such that $\theta_2 > O(\delta/\varepsilon)$ [for this, we can require $\mathrm{Es} > O(\varepsilon^{-1})$], then, owing to the temperature continuity at the interface, the estimate

$$\theta_1 \sim \theta_2 \tag{3.11}$$

should be valid. In this case, the temperature perturbations in the film and in the core are in the main order in condition (2.9), which implies that the thermodynamic processes in both areas are interrelated and offers the possibility of nonlocal terms occurring in the amplitude equation.

For the radial derivatives on the right-hand side of (2.8), we have

$$O(\mathrm{Es}\,\delta) < \theta_r(L) < O(\mathrm{Es}\,\delta\varepsilon^{-1}); \tag{3.12}$$

$$O(1) < \Theta_{0r}(L) < O(\varepsilon^{-1}). \tag{3.13}$$

As in [2], we can consider a situation in which hydrodynamic perturbations of the core and in the thin layer are in the main order in (2.8). The terms $(m/\varepsilon)w_{2y}$ and w_{1r} are the dominant contributions of the film and of the core, respectively. Owing to the condition $\mathrm{We} = O(\varepsilon^{-1})$, these quantities have the same order of magnitude.

Let us turn our attention to the kinematic condition. In the coordinate system moving along the z axis with velocity $W_{02}(1)$, relation (2.10) takes the form $l_t - (2/m)\,l\,l_z = u_2(y=1)$ (we omit higher-order terms). From this, we determine the time scale for the development of perturbations and also a relation between the parameters δ and ε for the evolution of the interface to be related to the processes inside the physical system considered:

$$\delta \sim \mathrm{We}\,\varepsilon^3. \tag{3.14}$$

The time dependence of the processes of perturbations evolution in the new system of coordinates is expressed in terms of $\tau = \delta t$.

4. Derivation of an Integro-Differential Amplitude Equation. Let

$$\mathrm{Re}_i = O(\varepsilon), \quad \mathrm{Pe}_1 = O(\varepsilon^{-1}), \quad \mathrm{Pe}_2 = O(1), \quad \mathrm{We} = O(\varepsilon^{-1}), \quad \mathrm{Es} = O(\varepsilon^{-3/2}), \quad \mathrm{Mn} = O(\varepsilon^{3/2}), \quad \mathrm{Dis}_i \leqslant O(\varepsilon^{-2}). \tag{4.1}$$

It follows from (3.1)–(3.6), (3.10), (3.11), and (3.14) that the asymptotic representation for the quantities takes the form

$$U_2 = \varepsilon^4 u + O(\varepsilon^5), \quad W_2 = W_{02} + \varepsilon^3 w + O(\varepsilon^4), \quad P_2 = P_{02} + p + O(\varepsilon),$$
$$\Theta_2 = \Theta_{02} + \varepsilon^{1/2}\theta + O(\varepsilon^{3/2}) \quad \text{in the film;}$$
$$U_1 = \varepsilon^2 u_c + \ldots, \quad W_1 = W_{01} + \varepsilon^2 w_c + \ldots, \quad P_1 = P_{01} + \varepsilon p_c + \ldots,$$
$$\Theta_1 = \Theta_{01} + \varepsilon^{1/2}\theta_c + \ldots \quad \text{in the core.} \tag{4.2}$$

Substituting (4.2) into (2.1)–(2.10) and taking into account (3.7)–(3.9), (3.12), and (3.13), we obtain the following problem for the main-order terms:
— in the thin layer

$$p_y = 0; \tag{4.3}$$
$$-p_z + (m/\overline{\mathrm{Re}}_1)w_{yy} = 0; \tag{4.4}$$
$$-u_y + w_z = 0; \tag{4.5}$$
$$\theta_{yy} = 0 \quad \text{for} \quad 0 < y < 1; \tag{4.6}$$
$$u = w = \theta = 0 \quad \text{for} \quad y = 0; \tag{4.7}$$

the conditions at the interface for $r = 1$ and $y = 1$ have the form

$$p = (\overline{\mathrm{We}}/\overline{\mathrm{Re}}_1)(\eta + \eta_{zz}); \tag{4.8}$$
$$u_{cz}(1) + w_{cr}(1) + m w_y(1) = \overline{\mathrm{Mn}}\,\theta_z(1); \tag{4.9}$$
$$\theta_{cr}(1) + \overline{q}\theta_y(1) = -\mathrm{E}\eta_z, \tag{4.10}$$

where $\mathrm{E} = \overline{\mathrm{Es}}\,TW_{0r}(1)$ [the value of T was given in the derivation of (3.10)];

$$u_c(1) = 0, \quad w_c(1) = 2(1 - 1/m)\eta; \tag{4.11}$$
$$\theta_c(1) = \theta(1); \tag{4.12}$$
$$\eta_\tau - (2/m)\eta\eta_z = u(1); \tag{4.13}$$

— in the core

$$\Delta u_c - (1/r^2)u_c = \overline{\mathrm{Re}}_1 p_{cr}; \tag{4.14}$$
$$\Delta w_c = \overline{\mathrm{Re}}_1 p_{cz}; \tag{4.15}$$
$$u_{cr} + (1/r)u_c + w_{cz} = 0; \tag{4.16}$$
$$\Delta\theta_c = 0 \quad \text{for} \quad 0 < r < 1; \tag{4.17}$$
$$u_c,\ w_c,\ \theta_c \quad \text{are bounded for } r = 0. \tag{4.18}$$

The corresponding parameters, each divided by its order, are denoted by $\overline{\mathrm{Es}}$, $\overline{\mathrm{Mn}}$, $\overline{\mathrm{We}}$, $\overline{\mathrm{Re}}_1$, and $\overline{q}$.

Problem (4.3)–(4.7) has the following solution:

$$p = p(z,\tau), \quad w = (\overline{\mathrm{Re}}_1/m)(p_z y^2/2 + A(z,\tau)y), \quad u = (\overline{\mathrm{Re}}_1/m)(p_{zz}y^3/6 + A_z y^2/2), \quad \theta = B(z,\tau)y. \tag{4.19}$$

Here $A(z,\tau)$, $B(z,\tau)$, and $p(z,\tau)$ are unknown functions that should be found from the boundary conditions.

We hence obtain $u(y=1) = (w_{yz}(y=1))/2 - p_{zz}\overline{\mathrm{Re}}_1/(3m)$, and expressing $w_y(y=1)$ in terms of (4.9), we have

$$u(y=1) = (\overline{\mathrm{Mn}}B_{zz} - u_{czz}(r=1) - w_{crz}(r=1))/(2m) - p_{zz}\overline{\mathrm{Re}}_1/(3m). \tag{4.20}$$

Using the Fourier transform $F(\theta_c) = \int\limits_{-\infty}^{\infty} \theta_c(r,z,\tau)\exp(-i\alpha z)\,dz$, we find from (4.17) with allowance for (4.18) that $F(\theta_c) = C(\alpha)I_0(\alpha r)$ in the Fourier space. In what follows, I_0 and I_1 are the modified Bessel functions of the zeroth and first order, respectively.

Let us determine the desired functions $B(z,\tau)$ and $C(\alpha)$ from conditions (4.10) and (4.12), which can be written as $C(\alpha)\alpha I_1(\alpha) + \overline{q}F(B) = -\mathrm{E}F(\eta_z)$ and $C(\alpha)I_0(\alpha) = F(B)$. Then

$$B(z,\tau) = -\mathrm{E}F^{-1}\{F(\eta_z)I_0(\alpha)/(\alpha I_1(\alpha) + \overline{q}I_0(\alpha))\}. \tag{4.21}$$

The hydrodynamic part of the problem in the core is solved similarly, but in a more complicated way (see [2]). If we introduce the stream function, find its Fourier image up to unknown functions, define these functions using the boundary conditions, and return by means of the inverse Fourier transform to the configuration space, we can find $\{w_{crz}(r=1) + u_{czz}(r=1)\}$ in (4.20).

Substituting the previously found expression (see formulas (26) and (27) in [2]), and also $p(z,\tau)$ from (4.8) and $B(z,\tau)$ from (4.21) into (4.20), we obtain from (4.13) that the required amplitude equation has the form

$$\eta_\tau + M_1\eta\eta_z + M_2(\eta_{zz} + \eta_{zzzz}) + \int\limits_{-\infty}^{\infty} T(\alpha) \int\limits_{-\infty}^{\infty} \eta_\zeta(\zeta,\tau)\exp(i\alpha(z-\zeta))\,d\zeta\,d\alpha$$
$$+ \int\limits_{-\infty}^{\infty} iG(\alpha) \int\limits_{-\infty}^{\infty} \eta(\zeta,\tau)\exp(i\alpha(z-\zeta))\,d\zeta\,d\alpha = 0, \tag{4.22}$$

where

$$M_1 = -2/m, \quad M_2 = \overline{\mathrm{We}}/3m, \quad T(\alpha) = -\overline{\mathrm{Mn}}\,\mathrm{E}\,\frac{1}{4\pi m}\,\frac{\alpha^2 I_0(\alpha)}{\alpha I_1(\alpha) + \overline{q}I_0(\alpha)},$$
$$G(\alpha) = \frac{1}{\pi m}\Big(1 - \frac{1}{m}\Big)\frac{\alpha^2 I_1{}^2(\alpha)}{\alpha I_1{}^2(\alpha) - \alpha I_0^2(\alpha) + 2I_0(\alpha)I_1(\alpha)}. \tag{4.23}$$

Remark 1. The case $\mathrm{Re}_i = O(1)$ is considered similarly. Only are Eqs. (4.14) and (4.15) changed in problem (4.3)–(4.18). The amplitude equation in this case has the form (4.22) and (4.23), and only the kernel $G(\alpha)$ is determined by another formula given in [2].

Remark 2. For $O(\varepsilon^{-1}) < \mathrm{We} < O(\varepsilon^{-2})$, the reduced problem (4.3)–(4.18) retains its form; only the terms u_{cz} and w_{cr} in (4.9) are omitted, and the second integral term with kernel $iG(\alpha)$ in (4.22) vanishes. In other respects, the amplitude equation is not changed.

5. Derivation of an Amplitude Equation in the Absence of the Effect of Perturbations in the Core on the Evolution of the Interface. Let

$$\rho_2/\rho_1 = O(1), \quad q = O(1), \quad \mathrm{Re}_i = O(\varepsilon) \quad \text{or} \quad \mathrm{Re}_i = O(1),$$
$$\mathrm{Pe}_i = O(1), \quad O(\varepsilon^{-2}) < \mathrm{We} < O(\varepsilon^{-1}), \quad \mathrm{Mn} = O(1).$$

The considerations of Sec. 3 can be repeated almost word for word, with a few exceptions. Estimate (3.10) now has the form $\theta_2 \sim \mathrm{Es}\,\delta\varepsilon$. The temperature and hydrodynamic perturbations are related by condition (2.8), with $\mathrm{Es} \sim \mathrm{We}$.

In the case considered, $\Theta_{01}(1+\delta\eta) - \Theta_{02}(1+\delta\eta) = O(\delta)$ and $\theta_2 > O(\delta)$ and, therefore, we assume that $\theta_1 \sim \theta_2$.

Instead of (3.7), (3.8), (3.14), and (3.15), we obtain

$$O(\delta) < w_r(L) < O(\mathrm{We}\,\delta\varepsilon), \quad O(\mathrm{We}\,\delta\varepsilon^2) < u_r(L) < O(\delta), \quad O(\mathrm{Es}\,\delta\varepsilon) < \theta_r(L) < O(\mathrm{Es}\,\delta), \quad \Theta_{0r}(L) = O(1).$$

The initial system (2.1)–(2.10) can then be simplified to the following reduced problem. Conditions (4.3)–(4.7) are satisfied in the thin layer. The conditions

$$\begin{gathered} p = (\overline{\mathrm{We}}/\overline{\mathrm{Re}}_1)(\eta + \eta_{zz}), \quad m w_y(1) = \mathrm{Mn}\ \theta_z(1), \quad q\theta_y(1) = -\mathrm{E}\ \eta_z, \\ u_c(1) = 0, \quad w_c(1) = 2(1 - 1/m)\eta, \quad \theta_c(1) = \theta(1), \quad \eta_\tau - (2/m)\eta\eta_z = u(1) \end{gathered} \tag{5.1}$$

are satisfied at the interface for $y = 1$.

To find the function $u(y = 1)$, it is necessary to know the dynamics only in the film, and, therefore, we do not formulate the problem in the core. By determining the required functions $A(z, \tau)$, $B(z, \tau)$, and $p(z, \tau)$ in (4.19) from boundary conditions (5.1) and substituting the quantity $u(y = 1)$ expressed as a functional of $\eta(z, \tau)$ into the reduced kinematic condition, we obtain that in this approximation the behavior of the interface is described by the equation

$$\eta_\tau + M_1\eta\eta_z + M_2(\eta_{zz} + \eta_{zzzz}) + M_3\eta_{zzz} = 0 \quad [M_1 = -2/m, \quad M_2 = \overline{\mathrm{We}}/3m, \quad M_3 = \mathrm{Mn}\,\mathrm{E}/2mq]. \tag{5.2}$$

Remark 3. The reduction method is apparently applicable also to the flow in a cylindrical pipe of circular cross section, which has a similar configuration.

6. On Possible Consequences of the Thermodynamic Effects Considered. The linear dispersion relation for the harmonics $\eta = \exp\{i\alpha z + \lambda\tau\}$ takes the form $\lambda = M_2(\alpha^2 - \alpha^4) - iG(\alpha) - i\alpha T(\alpha)$ for Eq. (4.22) and $\lambda = M_2(\alpha^2 - \alpha^4) + iM_3\alpha^3$ for Eq. (5.2). Thus, the effect of temperature perturbations in both cases is of a dispersive character.

It is known that dispersion effects can have a regularizing influence on the turbulent behavior of dynamic systems. For example, it has been shown in the numerical calculations of [6] that the chaotic character of the solution of the Kuramoto–Sivashinskii equation (1.1) changes abruptly when a term of the form $M_3\eta_{zzz}$ with the corresponding value of the coefficient M_3 is added to (1.1). The solutions become similar to a chain of impulses with equal amplitudes, which move as a whole, or to a sequence of solitary waves.

In accordance with the numerical calculations carried out in [2], the nonlocal term with the kernel $iG(\alpha)$, which is of a dispersive nature, also introduces some order into the chaotic character of the solutions of the Kuramoto–Sivashinskii equations. For example, motions with two characteristic spatial scales can occur: the periodic profile represents a long high crest with a small "hill" on it. The kernels $iG(\alpha)$ and $T(\alpha)$ are similar in structure, and the presence in the amplitude equation (4.22) of the nonlocal integral term with kernel $T(\alpha)$ due to thermodynamic effects that are active within the framework of the considered mechanism of perturbation development will probably yield the same results.

Equations (4.22) and (5.2) differ from the Kuramoto–Sivashinskii equation (1.1) only by the presence of terms of a dispersive character. The real parts of the dispersion relations for these equations are the same. This means that the energy of unstable long waves is transferred by the quadratically nonlinear term to attenuating short-wave perturbations. Such an organization ensures the boundedness of the solutions of the Kuramoto–Sivashinskii equation [7, 8]. This mechanism is very likely to lead to a similar boundedness of the solutions of Eqs. (4.22) and (5.2) as well, because the effects occurring in them, which are new in comparison with (1.1), are of a purely dispersive character for small perturbations.

These arguments and the numerical calculations for Eq. (5.2) of [6] and Eq. (4.22) with a single integral term with kernel $iG(\alpha)$ [2], which testify to the boundedness of the solutions found in space and time, along with the initial assumption that $\delta \ll \varepsilon$, allow us to expect that the thin layer in the cases considered is stabilized long before the perturbation amplitude of the interface becomes comparable with the thickness of this layer and, consequently, the film does not break up.

The specific character of possible secondary regimes described by the amplitude equations obtained requires further investigation.

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